

A physical application of Kerr-Schild groups

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September 19, 2002

Abstract

The present work deals with the search of useful physical applications of some generalized groups of metric transformations. We put forward different proposals and focus our attention on the implementation of one of them. Particularly, the results show how one can control very efficiently the kind of spacetimes related by a Generalized Kerr-Schild (GKS) Ansatz through Kerr-Schild groups. Finally a preliminar study regarding other generalized groups of metric transformations is undertaken which is aimed at giving some hints in new Ansätze to finding useful solutions to Einstein's equations.

1 Introduction

Kerr-Schild groups (or motions) are new kind of groups of metric transformations¹. In [1], their general structure as (local) Lie groups is worked out and several explicit solutions are solved. In another work, [2], a further study has been carried out to deal with the issue of their general resolution and existence. The results show that Kerr-Schild groups have a much richer structure than isometries or conformal symmetries. Some of the (new) features worth to be remarked are that they introduce a null vector field as the basic ingredient for the definition of a metric transformation, under some circumstances they contain infinite dimensional Lie algebras, or they allow for restricting isometries into subgroups according to their relation with respect to the null vector field. In this paper we shall start from the result that Kerr-Schild motions may be characterized by the system of differential equations (Kerr-Schild equations hereafter)

$$\mathcal{L}(\xi)g = 2h\ell \otimes \ell, \quad \mathcal{L}(\xi)\ell = b\ell, \quad (1)$$

¹The transformations are only required to be well behaved locally and the words “group” or “transformation” will stand for “local group” or “local transformation”.

where g is the fundamental tensor of a (Lorentzian) manifold, and ℓ is a real-valued one-form field. These objects are considered as data of the above system.² On the other hand, h and b are unknown C^∞ functions and $\vec{\xi}$ is the infinitesimal generator of a Kerr-Schild motion, also an unknown of the problem. Particularly, ξ is named a “Kerr-Schild vector field” (KS VF) in the same way that we have Killing or conformal vector fields.

In addition, other works in [2] we have tackled other new possibilities coming from a geometrical definition of metric transformations. That work has enough detailed and general results. However their physical application is not developed in accordance.

In this work we give some ideas that may prove to be useful towards such goal. Some of them are based upon physical applications of the two well-known metric symmetries, i.e. isometries and conformal symmetries. The main point here is that Kerr-Schild groups, or the new studied candidates, may yield relevant results beyond the ones of the two mentioned symmetries in some areas. Our aim is mainly to bring the attention to them.

In Sect. 2, some applications are put forward. Sect. 3 defines one of the presented ideas, focussing the attention on the Kerr-Schild case, because it is, as of now, the most studied situation in the literature. In Sect. 4 we apply it to geodesic Kerr-Schild groups (Kerr-Schild groups in which ℓ is geodesic). In Sect. 5, we begin with new candidates to metric transformations (or generalized metric symmetries) and give the results for the signature of the obtained space-times (following the idea of Sect.3) in order to discern which may be the most interesting cases to be further studied elsewhere. We end this paper with the Conclusions.

2 Some physical applications of generalized metric symmetries

As mentioned elsewhere, their use in physics is just starting. Therefore, we can only advance some ideas about their application in direct physical problems. These will be surely complemented in the future, but we shall show that there are already some hints. First, there are some works which implement usual metric symmetries (isometries and conformal symmetries) to Einstein’s equations, see for instance [3]-[9]. This implementation has yielded interesting results because the authors collected different theorems and propositions on the energy-matter content that were previously a part, as well as new results. But the most remarkable feature would be the direct way they were all treated together thanks to the use of isometries or conformal transformations. We believe that a similar work might prove to be as useful when based upon other sort of

²Only the direction of ℓ is relevant. Therefore, it is worth to refer to the congruence of null curves with tangent vector ℓ , say C_ℓ .

metric symmetries and will be dealt with elsewhere.

Secondly, even in the field of the problem of rigidity and elasticity in General Relativity, there appear some problems that can be written in terms of metric symmetries, or affine-like ones, see e.g. [10]-[13]. For instance, the Beltrami-Michel equations for any stress tensor, [12], can be obtained from the integrability conditions of a generalized metric symmetry. Moreover, each of the proposals that aim at generalizing the classical group of rigid motions in General Relativity could benefit from their analysis inside the frame of generalized symmetries. An example regards to the appearance of infinite dimensional algebras in our scheme which might be linked with the action of a certain group, or subgroup, of the generalized rigid motions. This will be the matter of subsequent work.

Finally, there is another possibility, closer to the main line of thought taken hitherto in the literature. The idea of using metric symmetries in order to group spacetimes is very well-established, specially for the case of isometries, see e.g. [8, 14]. In fact it is nowadays customary to use expressions such as G_3 , G_6 , ... Therefore, with the addition of new metric symmetries, we could begin to use similar classifications. Yet we shall focus on a refined idea:

Instead of just classifying spacetimes taking into account only inner metric symmetries, that would constitute a trivial extension of the usual isometric classification, we demand moreover that there exists some “external” relation among them. Of course, there are many alternatives for these “*ad hoc*” requirement, but we have decided that the very metric symmetry be the connection amongst such spacetimes. This is accomplished by demanding that the spacetimes that share the same algebra of a given metric symmetry must be linked to each other by finite relations of the same form as the considered local metric symmetry. The meaning of being of the same form will be defined in a moment.

This idea will be mainly applied to the Kerr-Schild case in this work because the (Generalized) Kerr-Schild (GKS) Ansatz has proven to be very useful in General Relativity, see e.g. [8, 15, 16]. The fact is that, although the same idea can be further applied without difficulty to other new metric symmetries, one should first discern which are the most interesting finite metric relations. One often asks for new Ansätze that would as well yield useful solutions to Einstein’s equations. However, their analysis turns out to be rather cumbersome for almost any alternative to conformal or GKS ones, see e.g. [17]. Now, we have the bonus that the structure of the final metrics ultimately rests on a symmetry problem, which, thanks to Lie’s theory on continuous groups, is entirely controlled in its linear and first order contributions, also gives the hope that other metric transformations, e.g. the ones in [2], will prove to be useful. We had not still carried out an exhaustive study for these much bigger set of new metric symmetries. Here we present only a basic step towards such goal in Sect. 5, which are the signatures of the resulting spacetimes and if they may change.

3 Application to Kerr-Schild groups

3.1 A preliminary remark

Among the results obtained in [1] we would like to stress the following one

Proposition 1 *If a spacetime is of a GKS form, i.e. $\tilde{g} = g + 2H\ell \otimes \ell$, the solution for its KSVFs is the same as for (g, ℓ) , i.e. Eqs. (1), where now $\tilde{b} = b$, $\tilde{h} = h + \mathcal{L}(\xi)H + 2bH$.*

It is important to notice that the set $\{\vec{\xi}\}$ is the same, but not necessarily its action on \tilde{g} , since, in general, $\tilde{h} \neq h$. This proposition will be useful in the following section and the examples later on.

3.2 Integration à la Kerr-Schild of a Kerr-Schild group

Definition 1 (Integration à la Kerr-Schild) *We will say that we have integrated à la Kerr-Schild some particular Kerr-Schild group whenever*

$$\exists H \mid \mathcal{L}(\vec{\xi})H + 2Hb + h \in \mathcal{F}(h), \quad \forall \vec{\xi}, \quad (2)$$

where H is a C^∞ -function of the manifold and $\mathcal{F}(h)$ is the set of functions that have the same functionality in the variables of the manifold as h has, and $\vec{\xi}$ is any KSVF of the considered Kerr-Schild group.

Consequently, from Proposition 1, and Eq. (2), $\tilde{g} \equiv g + 2H\ell \otimes \ell$ (where $\ell, b, \vec{\xi}, h$ are the same objects of the considered Kerr-Schild problem) will have not only the same Kerr-Schild group as g, ℓ , but also its action on \tilde{g} , g . The same could be straightforwardly extended to other kind of general metric symmetries. Let us recall that the notion seems by itself interesting: in order to study a particular finite metric relation, which is non-linear and, often, difficult to handle, use moreover the information given by its “corresponding” infinitesimal transformation, which is linear and of first order. Note that this analysis would not make any sense in the case of isometries, since the subset of metrics which are related by an isometric relation is simply the same metric itself.

Since ℓ is a null one-form field, only its direction is relevant in the previous definition, that is

Proposition 2 *The à la Kerr-Schild integration does not depend on the parametrization of \vec{l} .*

Proof: If the parametrization of the curves defining the vector field \vec{l} is changed, \vec{l} changes by a multiplicative factor, i.e. $\vec{l} \rightarrow \vec{l}' = A\vec{l}$, with $A \neq 0$ and henceforth $\vec{l} \rightarrow \ell' = A\ell$. Writing the problem of Kerr-Schild groups for two different parametrizations, eqs. (1), it turns out that

$$h' = h/A^2, \quad b' = b + \mathcal{L}(\vec{\xi}) \ln |A|, \quad (3)$$

where h and b are the solutions for a certain parametrization and $A (\neq 0)$ measures the change of parametrization between both cases. Then, eqs. (2) for ℓ' are $\tilde{h}' = h' + \mathcal{L}(\xi)H' + 2b'H' \in \mathcal{F}(h')$. The solution of these equations is clearly $H' = H/A^2$, where H is the solution to $A = 1$. Finally, writing $g' = g + 2H'\ell' \otimes \ell'$, one gets $g' = g + 2H\ell \otimes \ell$. Thus, the only characteristic property of ℓ is its direction, as remarked elsewhere.

We now proceed to solve the most representative examples worked out in [1] as well as a new one, which is the general solution of Kerr-Schild groups for Kerr-Newman spacetimes taking ℓ to be one of its principal null directions, Example 2. The reason of the subdivision in the presented examples is to be found in the results on the study of the general existence of Kerr-Schild groups, [2]. The non-geodesic case is left aside since the obtained solutions in [2] do not have apparently an immediate physical use.

4 Geodesic ℓ with $\Delta \neq 0$

To begin with we define the *scalar* Δ as:

$$\Delta := -2D\theta + 4\theta^2 - 3\bar{R} + 2l^\mu \nabla_\sigma \nabla^\sigma l_\mu + DM - 2M\theta \neq 0, \quad (4)$$

where

$$\theta = \frac{1}{2} \nabla_\rho l^\rho, \quad D := l^\rho \nabla_\rho, \quad \bar{R} = R_{\rho\mu} l^\rho l^\mu, \quad (5)$$

and M is defined by ℓ geodesic $\iff \vec{a} \equiv D\vec{l} = M\vec{l}$.

We include it here because this scalar appears as a fundamental quantity in the study of the general existence of Kerr-Schild groups. However, in the present work, it is only necessary to know that it classifies the possible Kerr-Schild groups generated by geodesic ℓ , according to whether Δ vanishes or not..

Example 1 (Radial and spherically symmetric ℓ in flat spacetime) *The integration à la Kerr-Schild for any geodesic, radial, spherically symmetric null vector field of flat spacetime yields the set of spacetimes linked by $g = \eta + 2H\ell \otimes \ell$.*

This first example of this section comes from flat spacetime, but one would have attained the same result starting from any of the resulting g .

Proof: The Kerr-Schild equations read

$$\mathcal{L}(\xi)\eta = 2h\ell \otimes \ell, \quad \mathcal{L}(\xi)\ell = b\ell. \quad (6)$$

The solution to this problem taking $\ell = \frac{1}{\sqrt{2}}d(t \pm r)$, where we use standard spherical coordinates

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

is shown to be $h = 0$, $b = 0$ and $\{\vec{\xi}\} = SO(3) \otimes T_t$, where T_t stands for translations along the t axis.

The equations that we need to solve are then

$$\mathcal{L}(\xi)H = 0 \quad (7)$$

whose solution is $H = H(r)$. Thus, the collection of metrics related by a GKS transformation that have $SO(3) \otimes T_t$ as a solution to their own spherical Kerr-Schild problem is

$$\tilde{g} = \eta + 2H(r)\ell \otimes \ell, \quad (8)$$

with $\ell = 1/\sqrt{2}(dt \pm dr)$.

The set of metrics read in these coordinates (we focus on the case $\ell = (1/\sqrt{2})(dt + dr)$, though the same spacetimes are obtained for $\ell = (1/\sqrt{2})(dt - dr)$)

$$ds^2 = -(1 - H)dt^2 + 2Hdt dr + (1 + H)dr^2 + r^2 d\Omega^2, \quad H = H(r). \quad (9)$$

A general cobasis, which avoids coordinate problems at the possible horizons, is

$$\begin{aligned} \Theta^0 &= (1 - \frac{H}{2})dt - \frac{H}{2}dr, & \Theta^1 &= (1 + \frac{H}{2})dr + \frac{H}{2}dt, \\ \Theta^2 &= r d\theta, & \Theta^3 &= r \sin \theta d\varphi. \end{aligned} \quad (10)$$

One can recover for $H < 1$ the explicit static expression of the metric by making the well-known change of the t coordinate given by $dt = dt' + [H/(1 - H)] dr$, where t' is a new coordinate, whereas r , θ , and ϕ remain unchanged. In this set of coordinates one gets

$$ds^2 = -(1 - H)(dt')^2 + (1 - H)^{-1}dr^2 + r^2 d\Omega^2, \quad H = H(r). \quad (11)$$

We change in the following the name of t' to t , the usual Schwarzschild time coordinate. The natural cobasis to study all these metrics in the region $H < 1$ is then $\Theta^0 = \sqrt{1 - H} dt$, $\Theta^1 = (1/\sqrt{1 - H})dr$, $\Theta^2 = r d\theta$, $\Theta^3 = r \sin \theta d\varphi$.

The Riemannian tensor has the following non-zero components (and, in addition, the ones obtained by index symmetries). The expressions are equally valid for both cobasis.³

$$\begin{aligned} R_{0101} &= -H''/2, & R_{0202} &= R_{0303} = -H'/2r, \\ R_{1212} &= R_{1313} = H'/2r & R_{2323} &= H/r^2. \end{aligned} \quad (12)$$

The Ricci tensor has the following non-zero components

$$R_{00} = -R_{11} = -\frac{1}{2} \left(H'' + \frac{2H'}{r} \right), \quad R_{22} = R_{33} = \frac{1}{r} \left(H' + \frac{H}{r} \right). \quad (13)$$

³This result can be extended to any other changed cobasis in any of the two two-planes expanded by $dt - dr$ and $d\theta - d\varphi$.

And the scalar curvature is given by

$$R = H'' + \frac{4H'}{r} + \frac{2H}{r^2}. \quad (14)$$

Finally, their Einstein tensor has the following expression

$$G_{00} = -G_{11} = \frac{1}{r} \left(H' + \frac{H}{r} \right), \quad G_{22} = G_{33} = -\frac{1}{2} \left(H'' + \frac{2H'}{r} \right). \quad (15)$$

Among the (V_4, \tilde{g}) one finds Schwarzschild vacuum and interior solution, Reissner-Nordström and de Sitter solutions, and many others. Particularly, these are obtained with $(G = c = 1$ throughout this work) $H = 2m/r$ (the only non-flat vacuum solution within the above set), $H = 2m/r - Q^2/r^2$, $H = (\Lambda/3)r^2$, respectively, and where m is the mass of the object, Q its charge and Λ is the cosmological constant. The only case going back to flat spacetime is the trivial one $H = 0$. In order to show the usefulness of the preceding results, we will say that they have been applied very recently to study (quantum) regular interiors for black holes, see e.g. [18]-[20].

Besides these conclusions, it is worth emphasizing that the appearance of the de Sitter's solution has been in the form of its *causal connected part*. Thus this result may add some hints about the role played by ℓ (this result must be related to the fact that ℓ is a null one-form) in the Kerr-Schild transformation and its physical meaning, see also [21]. Results which, on the other hand, may as well be useful for the new situations depicted in Sect. 5, in order to relate the desired physical properties of the final solutions with the type of transformation one has to use for.

Example 2 (Kerr-Newman spacetimes) *The integration à la Kerr-Schild for any of the principal null directions of Kerr-Newman spacetimes is given by $g = \eta + 2H(r, z)\ell \otimes \ell$, if ℓ is rotational and by the result of Example 1 for the case of an irrotational ℓ .*

The first thing to do is to solve the Kerr-Schild groups in Kerr-Newman spacetimes for their principal null directions. The result is

Lemma 1 *The Kerr-Schild groups for Kerr-Newman spaces associated with the principal null directions are given by $T_t \otimes T_\phi$ for rotational ℓ , and by $T_t \otimes SO(3)$ for the irrotational case.*

Here t and ϕ are Kerr coordinates, not Boyer-Lindquist coordinates. This result is remarkable because it extends Kerr-Schild groups to the most useful models of charged, spinning objects.

Proof: The path of solving eqs. (2) directly is unnecessarily long. Instead we can make use of Proposition 1.

In our case, it is well-known that Kerr-Newman spacetimes can be written in a Kerr-Schild form, i.e. $g = \eta + 2H\ell \otimes \ell$, where ℓ is any of its principal null

directions, and $H = H(r, z)$, see e.g. [8, 22], yet its expression is irrelevant to our aims. The consequence—using Proposition 1—is that the Kerr-Schild problem for Kerr-Newman spacetimes choosing ℓ to be one of its principal null directions can be reduced to the resolution of Eqs. (6), where ℓ has the expression

$$\ell = \frac{1}{\sqrt{2}} \left(dt + \frac{z}{r} dz + \frac{rx + ay}{r^2 + a^2} dx + \frac{ry - ax}{r^2 + a^2} dy \right), \quad (16)$$

where a is now simply a constant, (its meaning as the spin of a particle belongs to Kerr-Newman spacetimes) and $\{t, x, y, z\}$ are now cartesian coordinates of the flat spacetime. It is easy to show that ℓ is geodesic and shear-free. If a is zero, we recover the radial null vector of Example 1. If $a \neq 0$, ℓ is rotational. The function r is defined by $r^2(x^2 + y^2 + z^2) + a^2 z^2 = r^2(r^2 + a^2)$ as in Kerr-Newman spacetimes and the $1/\sqrt{2}$ factor is introduced in order to link the irrotational situation with the radial ℓ of Example 1.

Again, a direct resolution of Eqs. (6) is rather long, though easier than a direct resolution within Kerr-Newman spacetimes. Nevertheless, we shall make use of a—non-trivial—result about the general resolution of Kerr-Schild groups. When ℓ is geodesic, it can be shown that Δ determines the type of solution to the problem of Kerr-Schild groups. Particularly, if $\Delta \neq 0$, KSVFs are the solutions of the system

$$\mathcal{L}(\xi)\gamma = 0, \quad \mathcal{L}(\xi)\ell = b\ell, \quad (17)$$

where γ is a rang-two symmetric tensor, whose exact expression is unnecessary now, see [2] for details. It suffices to know that, in our case, we have the bonus that $\gamma = \eta$. Thus our problem reduces to a problem of isometries *restricted* by the second set of equations in (17).

In the same reference, one can find the conditions under which Δ may vanish in, e.g., flat spacetime. A condition is that ℓ cannot be rotational. Therefore for $a \neq 0$, $\Delta \neq 0$. For the irrotational case, $a = 0$, we can directly compute Δ since the expressions are rather simple. The result is $\Delta = 1/r^2 \neq 0$. Thus $\Delta \neq 0$ for the principal null directions of Kerr-Newman metrics, regardless the value of a .

The final step is to solve Eqs. (17). The solutions must be a linear combination of the infinitesimal generators of the Poincaré group that satisfy the second group of equations of (17). The calculations are given in [2]. The end result is that $h = b = 0$, and $\{\tilde{\xi}\} = \{\partial_t, x\partial_y - y\partial_x\}$ for the case $a \neq 0$, and $\{\partial_t, x\partial_y - y\partial_x, x\partial_z - z\partial_x, y\partial_z - z\partial_y\}$. This is the solution for flat spacetime. The infinitesimal generators are the same for Kerr-Newman spaces as we know from Proposition 1. We can also use it in order to find their action on them. The result is, in any case, $\tilde{h} = \tilde{b} = 0$. Therefore, the KSVFs are the Killing vectors of Kerr-Newman spaces. This finishes the proof of Lemma 1.

The final step is to solve Eqs. (2). The solution is: for the rotational case, $H = H(r, z)$ and for the irrotational case, $H = H(r)$.

We note that a restriction of the spacetimes corresponding to the rotational case has been recently used in [23] in order to deal with the problem of a (quantum) source origin for the (charged) spinning particle within supergravity and string fields for the source. The new spacetimes are chosen from a direct generalization of Kerr-Newman spacetimes, letting the mass function to have a free function of r . We remark that the whole family, above presented, has not been used yet. We reckon that taking it into account, and using the local irrotational set of observers in order to set the physical conditions near the core might yield new results—recall the results in the radial (irrotational) case before—, specially for the case where vacuum polarization is the dominant effect.

Finally, let us recall that the main idea behind the GKS Ansatz is to choose a particular seed metric, say g_0 , and an ℓ satisfying some physical requirements. This procedure is still present in Kerr-Schild groups because g and ℓ (its direction) are the data of the problem. One is thus encouraged to perform a further study for other geodesic ℓ with $\Delta \neq 0$. For instance, we point out that Vaydia and Kerr-Vaydia metrics, [24], are spacetimes which could benefit from Kerr-Schild groups for the obtention of some hints towards interior solutions. Moreover, another option could start from the solution of Kerr-Schild symmetries for static spherically symmetric spacetimes given in [1]. Besides the “ $\rho + p = 0$ ” family, the rest of interior stellar models would be equally subgrouped according to their shared Kerr-Schild symmetries which should lead to similar results to those of [18, 20]. Finally, the connection with cosmological issues can be started from [25].

4.1 Geodesic ℓ with $\Delta = 0$

This is the last section on KSVFs. The null vector fields which are geodesic and satisfy $\Delta = 0$ are still a matter of further research. For the time being, it suffices to know that in spacetimes where \vec{l} is a principal null direction and $\vec{R} = 0$ the family of such vectors includes the “cylindrical” and “cartesian” (or “parallel”) cases, see below. Both cases constitute paradigms of Kerr-Schild groups. For the sake of brevity, we shall consider flat spacetime in the examples, although an extension of the results to several spacetimes of physical interest, such as those representing pp -waves, is readily accomplished (see later). Using cylindrical coordinates, the line-element of flat spacetime takes the form

$$ds^2 = -dt^2 + d\rho^2 + \rho^2 d\varphi^2 + dz^2. \quad (18)$$

The “cylindrical” ℓ corresponds to $\ell_{(\pm)} = (1/\sqrt{2})(\pm dt + d\rho)$. In the second case, using cartesian coordinates,

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2, \quad (19)$$

whence one defines $\ell_{(\pm)} = (1/\sqrt{2})(\pm dt + dx)$ as a “cartesian” (or “parallel”) ℓ (of course, x may be interchanged by either y or z).

Both ℓ yield rather particular solutions to the problem of Kerr-Schild groups. In the first case, we find (see [1]) a local group which must be reduced when considering global topological properties of flat spacetime, whereas in the cartesian case, the Lie algebra turns out to be infinite dimensional, i.e. containing some *functional* freedoms.

Due to these special properties, it is worth studying them here. The results are

Example 3 *The à la Kerr-Schild integration of the cylindrical ℓ in flat space-time yields the spacetime defined by the metric tensor $g = \eta + 2H(\rho)\ell \otimes \ell$, where $\ell = (1/\sqrt{2})(\pm dt + d\rho)$, and $\{t, \rho\}$ are usual cylindrical coordinates of cylindrically symmetric spacetimes.*

The calculation of this result follows similar steps as those of the preceding examples. The expressions for the KSVFs can be read from [1]. We note that the result above is valid either for the local group and for the global one.

The geometrical properties of this case can be worked with the aid of two different orthonormal cobasis as before. An analogous change in the time coordinate t as that of the previous section allows us to write them in an explicit static form if $H < 1$. The expressions are analogous to (11).

The Riemann tensor components are ⁴ ($H = H(\rho)$, and $A' \equiv dA/d\rho$)

$$R_{0101} = -H''/2, \quad R_{0202} = R_{0303} = -H'/2\rho, \quad R_{1212} = H'/2\rho, \quad (20)$$

and the rest are obtained by index permutation or are zero. The Ricci tensor for these metric-spaces has the following non-zero components

$$R_{00} = -R_{11} = -(1/2)[H'' + (H'/\rho)], \quad R_{22} = H'/\rho. \quad (21)$$

Whence the scalar curvature is given by $R = H'' + (2H'/\rho)$.

Finally, their Einstein tensor has the following non-zero components.

$$G_{00} = -G_{11} = H'/2\rho, \quad G_{22} = -H''/2, \quad G_{33} = G_{22} + 2G_{11}. \quad (22)$$

The addition of a cosmological constant does not change the relation between T_{00} and T_{11} , though the relation among the spatial pressures allows for changes in the signs of T_{22} and T_{33} . The situation is now different with respect to Example 1, where we had well-known solutions for the classical vacuum case, or the case of electromagnetic fields. We have now an axis of symmetry. The only vacuum solution is now flat spacetime, which is obtained with $H = \text{const.}$. The solution for $R = 0$ is $H = a + b/\rho$, where a, b are constants (a is a gauge freedom), and corresponds to the electromagnetic field of a linear charged distribution, located along the axis $\rho = 0$, and is an analogue of the Reissner-Nordström solution for the symmetry we are considering.

⁴The results are again the same for several cobasis, including the “regular” and the “static” ones.

We shall not analyze here their geometric properties (singularities, possible extensions, etc). It seems that no well-established physical system can be attached to such relations and values of pressure and density.

Example 4 *The à la Kerr-Schild integration of the cartesian (plane) ℓ in flat Spacetime can not be accomplished.*

This result comes from the fact that the Lie algebra is now infinite dimensional as we will now show. For $ds^2 = -2du dv + dx^2 + dy^2$, and $\ell = du$, where $u := 1/\sqrt{2}(t+z)$, $v := 1/\sqrt{2}(t-z)$, being t, x, y, z standard cartesian coordinates of flat spacetime, the system of Eqs. (6) has the following solution:

$$\begin{aligned} h &= -\ddot{A}x - \ddot{B}y - \dot{C} + \ddot{D}v, \quad b = \dot{D} \\ \vec{\xi} &= D\partial_u + [-\dot{F}v + \dot{A}x + \dot{B}y + C]\partial_v + (-dy + A)\partial_x + (dx + B)\partial_y, \end{aligned} \quad (23)$$

where the functions A, B, C, D are arbitrary C^∞ functions of the variable u , $\dot{A} \equiv dA/du$, etc., and d is a real constant. It is the presence of these free functions that tells us that the associated Lie algebras are infinite dimensional.

Consequently, the integrating system is now

$$\mathcal{L}(\vec{\xi})H + 2H\dot{D} = \tilde{h} - h, \quad (24)$$

where $\tilde{h} = -(\tilde{A})''x - \dots$, changing all functions and constants by their respective expressions with tilde.

Notice that the solution has to be valid for any combination of the unknown functions and constants of the Kerr-Schild solution. The end result is that only the trivial solution $H = 0$ satisfies Eqs. (24) and therefore Definition 1.

Notice that from Proposition 1 the results above can be extended to include any spacetime with $g = \eta + F(u, v, x, y) du \otimes du$. Particularly, if $\partial F/\partial v = 0$ we have the pp-waves metrics, see e.g. [8, 26].

This result does not mean that for any subgroups the à la Kerr-Schild integration exists, e.g. for any unidimensional subgroup. However, it is enough, as of now, to remark the impossibility of fulfilling Definition 1 for some general (intrinsic) Kerr-Schild algebras. On the other hand, it yields not only a limit for the grouping that can be attained with Definition 1 itself —forbidding too free solutions—, but it might also help understanding some solutions to Einstein's equations from a different viewpoint. For instance, pp-waves and flat spacetime do *not* share their KSVFs, whereas Kerr-Newman and flat spacetimes *do*. A similar result was obtained twenty-five years ago by L. Defrise-Carter [27] regarding the “isometrization of conformal symmetries. All this adds a new argument to the conclusion that although Kerr-Newman and pp-waves are Kerr-Schild metrics, their relation with flat spacetime is completely different.

5 First steps towards finding other metric symmetries

We will now introduce other metric symmetries candidates. In [2] some new candidates of metric symmetries were proposed and some examples were presented. The examples were mainly intended to prove the existence of Lie algebras for each candidate. The aim here is to give additional arguments in favor of their study. We thus think that a knowledge of some of their physical consequences could help deciding which are worth to be studied in more detail. A necessary point to be considered is the signature change that may happen when using such new metric relations. In the Kerr-Schild situation this issue was absent because a contribution of the type $\ell \otimes \ell$ with $\ell \cdot \ell = 0$ does not change its signature. For the rest of the situations —excluding isometries— the signature of the final metric may change. Let us finally recall the point stressed in the introduction that a non-Lorentzian signature may be of interest if, e. g., we aim at dealing with problems where space-like metrics are the relevant object, or where signature change plays a definite role.

Of course, for the sake of brevity, we refer the reader to [2] for details on these metric symmetries.

5.1 The ℓ - m candidates

This set is built upon the metric symmetries that a pair of null one-form fields, say ℓ and m with $\ell \wedge m \neq 0$, can create and is related with the light cone structure of spacetime. Obviously, a subcase would be that of Kerr-Schild groups developed before, where only one null vector field is used.

The associated finite relations read

$$\tilde{g} = g + H \ell \otimes \ell + G (\ell \otimes m + m \otimes \ell) + F m \otimes m, \quad (25)$$

where F, G, H are C^∞ functions on the manifold. The study of the signature of \tilde{g} yields, see also [28] ⁵

$\text{sign}(H) = \text{sign}(F) = 1$	$4HF > (1+G)^2, \quad E,$ $4HF = (1+G)^2, \quad D,$ $4HF < (1+G)^2, \quad L.$
$\text{sign}(H) = \text{sign}(F) = -1$	$4HF > (1+G)^2, \quad A,$ $4HF = (1+G)^2, \quad D,$ $4HF < (1+G)^2, \quad L.$
$\text{sign}(H) \neq \text{sign}(F)$	$L.$
$H = 0$ or $F = 0$	$L.$ For $G = -1, \quad D.$
$H = 0$ and $F = 0$	$L.$ For $G = -1, \quad \text{double } D.$

⁵We shall choose a Lorentzian signature for g , particularly, $g = \text{diag}(-1, 1, 1, 1)$. Of course, similar results are valid if one starts with the $g = \text{diag}(1, -1, -1, -1)$.

Here (E)uclidean, (D)egenerated, (L)orentzian, (A)rtinian and double (D)e-generated signatures mean $(1, 1, 1, 1)$, $(0, 1, 1, 1)$, $(-1, 1, 1, 1)$, $(-1, -1, 1, 1)$, $(0, 0, 1, 1)$ respectively.

5.2 The p - q candidates

This set is a similar one with respect the previous Section. The difference lies now in that the metric symmetries are built using a pair of space-like one-form fields, say \mathbf{p} and \mathbf{q} with $\mathbf{p} \wedge \mathbf{q} \neq 0$. Their corresponding finite transformations are

$$\tilde{g} = g + H \mathbf{p} \otimes \mathbf{p} + G (\mathbf{p} \otimes \mathbf{q} + \mathbf{q} \otimes \mathbf{p}) + F \mathbf{q} \otimes \mathbf{q}. \quad (26)$$

The study of the signature of \tilde{g} yields

$\text{sign}(H+1) = \text{sign}(F+1) = \pm 1$	$G^2 > (1+H)(1+F), \quad A.$ $G^2 = (1+H)(1+F), \quad D.$ $G^2 < (1+H)(1+F), \quad L,$
$\text{sign}(H+1) \neq \text{sign}(F+1)$	$A.$
$H = -1$ or $F = -1$	$L.$ For $G = 0,$ $D.$
$H = -1, F = -1, G = 0$	double D.

In this case Artinian, Degenerated and double Degenerated signatures refer to $(-1, 1, 1, -1)$, $(-1, 1, 1, 0)$ and $(-1, 1, 0, 0)$, respectively.

5.3 The u - p candidates

This last set is built upon the metric symmetries that a time-like one-form field and a space-like one, say \mathbf{u} and \mathbf{n} with $\mathbf{u} \wedge \mathbf{n} \neq 0$ can create. Of course, combinations of this group correspond to different cases of Section 5.1. Nevertheless, there some problems where it is easier to work directly with a time-like vector field directly. For instance, in a “time-like” version of the GKS Ansatz, that is, if one choses a four velocity vector field instead of a null one. In this case, the corresponding same-type finite relations are

$$\tilde{g} = g + H \mathbf{u} \otimes \mathbf{u} + G (\mathbf{u} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{u}) + F \mathbf{n} \otimes \mathbf{n}. \quad (27)$$

The study of the signature of \tilde{g} yields

$\text{sign}(H - 1) = \text{sign}(F + 1) = 1$	$G^2 > (H - 1)(1 + F), \quad L,$ $G^2 = (H - 1)(1 + F), \quad D,$ $G^2 < (H - 1)(1 + F), \quad E.$
$\text{sign}(H - 1) = \text{sign}(F + 1) = -1$	$G^2 > (H - 1)(1 + F), \quad L,$ $G^2 = (H - 1)(1 + F), \quad D,$ $G^2 < (H - 1)(1 + F), \quad A.$
$\text{sign}(H - 1) \neq \text{sign}(F + 1)$	$L.$
$H = 1$ or $F = -1$	$L.$ For $G = 0,$ $D.$
$H = 1, F = -1, G = 0$	double D.

Now the degenerated cases are analogous to the ones presented in the two previous cases. In this set one finds some situations of interest, as the one having a double four-velocity field deformation, which should be worked with more detail, see also [21].

6 Conclusions

We have devoted this work to develop a particular application of metric symmetries aiming at finding new solutions of Einstein's equations through a major control on geometrical elements of the final solutions. For instance, in the more considered case of Kerr-Schild groups, the geometrical object is a null vector field which has a clearly physical interpretation, as well. This method may open a new path to finding useful solutions to Einstein's equations. This has been shown along the text with several examples, including the resolution of Kerr-Schild groups for Kerr-Newman spacetimes for their principal null directions, which constitute the basis for current studies on the structure of black holes, or spinning particles. Moreover, this method may prove to be a complement of the usual (G)KS transformations, which have been considered always from a purely finite viewpoint only. The consequences of using Kerr-Schild groups are still to be explored in other spacetimes as the ones considered here as remarked elsewhere.

For the other cases, i.e., other new metric symmetries, the interpretations are still to be worked out to achieve a similar status to that of Kerr-Schild symmetries. We have just set the basic pieces towards that goal, namely, a study regarding the signature change that may happen in the corresponding metrics. A worth control will only be gained if, at the same time, an extensive study of these new metric symmetries is carried out regarding the structure of their Lie algebras or, at least, by finding out some relevant examples.

Finally, let us recall that other possible applications, mentioned in the introduction, i.e. that of a implementation of general metric symmetries to study energy-matter properties of spacetime following the line of thought of [4, 5, 7]

and the approximation to the rigidity problem, [10]-[13], will doubtless fill in most of the gap towards the physical comprehension of metric symmetries. Therefore, besides the mentioned general studies, these are as well desirable areas to be further considered. All this might lead to a feed back reaction between the algebraical and physical issues of the fabric of metric symmetries.

Acknowledgements

I am indebted with Emili Elizalde, who has help me in the way of relating Kerr-Schild symmetries with physical applications through enlightening comments and Bartolomé Coll who proposed me to consider other candidates of metric motions a part from the Kerr-Schild ones. Alexander Ya. Burinskii is acknowledged for stimulating discussions and providing me his work prior to publication. S. R. H. wishes to thank the direcció General de Recerca, Generalitat de Catalunya for financial support.

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